CSCI 2244 – Homework 8

Out: Friday, November 8, 2019
Due: **Saturday**, November 16, 2019, 11:59pm

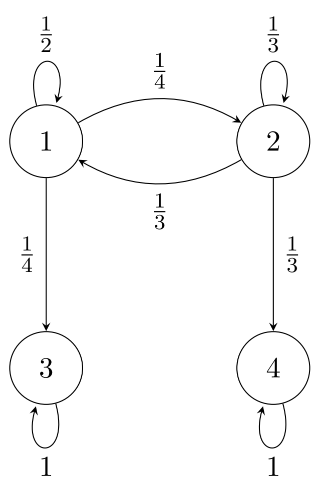
This homework consists of written exercises. You *must* type your solutions. See the "Assignments" section in the syllabus for advice about doing this. You should submit your homework via Canvas. In particular, you should upload a pdf file called:

`FirstName_LastName_Homework8.pdf`

Please use your full first name and last name, as they appear in official university records. The reason for doing so is that the TAs and I must match up these names with the entries in the gradebook.

1 Variances of Markov Chain Passage and Absorption Times

Task 1.1 (12 pts). Consider the following Markov chain (it is the same as the one from homework 7):

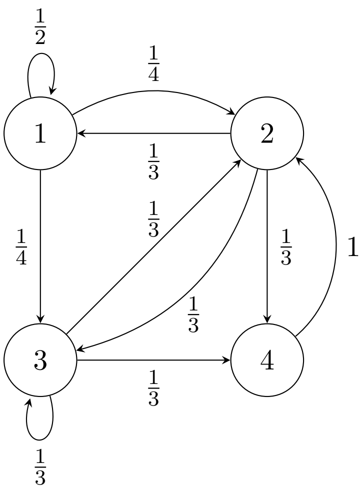


Answer the following questions assuming we start in the state specified and repeatedly transition until reaching an absorbing state:

(a) Starting from state 1, what is the variance of the number of times the chain will be in state 2 before absorption?

(b) Starting from state 2, what is the variance of the number of transitions we need to take to reach an absorbing state?

Task 1.2 (12 pts). Consider the following Markov chain (it is the same as the one from homework 7):



(a) What is the variance of the recurrence time for state 4?

2 Variation Distances

Recall that variation distance between two distributions D_1 and D_2, written $\|D_1 - D_2\|$, gives a measure of how different two distributions are. Given a Markov chain T, a number n and a state x, we wrote T_x^n for the distribution of states after n transitions given that the starting state is x. Assuming T is a regular chain with stationary distribution w, we are interested in the question of bounding $\|T_x^n - w\|$ in order to understand how far away the chain is from its stationary distribution after n iterations. We defined

$$\Delta(n) = \max_x \|T_x^n - w\|$$

which is the worst case distance over all possible starting states. The following theorem can be used to bound $\Delta(n)$:

Theorem 1. Let T be as above. Let m_j be the smallest entry in the jth column of T, and set $m = \sum_j m_j$. Then for all n:

$$\Delta(n) \leq (1 - m)^n$$

Task 2.1 (8 pts).

(a) Consider the transition matrix

$$R = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

First, try applying to theorem to R. Why doesn't it say anything useful? Now, apply the theorem to R^2. Relate the bound you obtain on the behavior of R^2 to a bound on $\Delta(n)$ for the original matrix R. (Hint: remember that the variation distance is non-increasing, so $\Delta(n+1) \leq \Delta(n)$.)

(b) Consider the transition matrix

$$Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Apply the theorem to Q and Q^2. As above, translate the resulting bound for Q^2 into a bound on $\Delta(n)$ for the original matrix Q. Which method leads to a better bound on $\Delta(n)$ for Q? (Careful: make sure you take into account that 1 iteration of Q^2 corresponds to two transitions of Q)